$\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$

$\vec{Y}'(t) = \begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \vec{F}(t, \vec{Y}(t)) \in \mathbb{R}^n$

Sec 4.9: Euler's Approximation Method for Systems of Differential Equations.

Suppose $\vec{Y}(t)$ is a the solution to the 1st order system:

$$\vec{Y}\,' = F(t, \vec{Y}), \quad \vec{Y}(t_0) = \vec{Y}_0$$

We can extend the Euler's Approximation Method to approximate $\vec{Y}(t^*)$ as follows:

- For a given step size $h = (t^* - t_0)/n$, define the grid points $t_0, t_1, \cdots, t_n = t^*$ by the formula $t_k = t_0 + kh$ where $0 \le k \le n$.
- Compute $\vec{Y}_{k+1} \approx \vec{Y}_k + h \cdot F(t_k, \vec{Y}_k)$ where $0 \le k \le n-1$.
- $\vec{Y}_n \approx \vec{Y}(t^*)$.

Ex1 [Fall 2013] You solve the initial value problem $y_1' = 3 + y_2^2, y_2' = 4t - y_1, y_1(1) = 1, y_2(1) = 3$ using the Euler method with $h = 0.05$. Then the approximation you find for $\vec{Y}(1.05)$ is:

$\vec{Y}\,' = F(t, \vec{Y}) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 + y_2^2 \\ 4t - y_1 \end{pmatrix}$

(a) $y_1 = 1.6, y_2 = 3.15$

(b) $y_1 = 1.6, y_2 = 3.16$

(c) $y_1 = 1.6, y_2 = 2.95$

(d) $y_1 = 2.2, y_2 = 3.3$

h

t_0 $\quad t^* = 1.05$

$\vec{y}(t_1) = \vec{y}(0.05) \approx \vec{Y}_1 = \vec{Y}_0 + 0.05\,\vec{F}(t_0, \vec{Y}_1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0.05 \begin{pmatrix} 3 + 3^2 \\ 4.5 - 1 \end{pmatrix} = \begin{pmatrix} 1.6 \\ 3.15 \end{pmatrix}$

DNF

$Y_0 = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$h = 0.05$

Ex2 [Spring 2015] You solve the initial value problem $y_1' = y_2 + t, y_2' = y_1 + 1, y_1(0) = 1, y_2(0) = 3$ using the Euler method with $h = 0.1$. Then the approximation you find for $\vec{Y}(0.2)$ is:

$\vec{F}(t, \vec{y}) = \vec{Y}\,' = \begin{pmatrix} y_2 + t \\ y_1 + 1 \end{pmatrix}$ $\quad Y_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(a) (1.6, 3.4)

$h = 0.1$

(b) (1.65, 3.431)

$n = 2$

(c) (1.62, 3.43)

(d) (1.63, 3.43)

$\vec{y}(t_1) = \vec{y}(0.1) \approx \vec{Y}_1 = \vec{Y}_0 + 0.1\,\vec{F}(t_0, \vec{Y}_0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0.1 \begin{pmatrix} 3 + 0 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} 1.3 \\ 3.2 \end{pmatrix}$

$\vec{y}(t_2) = \vec{y}(0.2) \approx \vec{Y}_2 = \vec{Y}_1 + 0.1\,\vec{F}(t_1, \vec{Y}_1) = \begin{pmatrix} 1.3 \\ 3.2 \end{pmatrix} + 0.1 \begin{pmatrix} 3.2 + 0.1 \\ 3 + 1 \end{pmatrix} = \begin{pmatrix} 1.63 \\ 3.43 \end{pmatrix}$